Series Tests

1. **Geometric Series.** The series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.

2. **Divergence Test.** If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. **Integral Test.** If $a_n = f(n)$ where f is a continuous, positive, decreasing function, then $\int_1^{\infty} f(x)\,dx$ and $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

4. **p-Series.** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

5. **Comparison Test.** If $a_n \geq b_n \geq 0$ for all n from some point on, then
   - if $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} b_n$.
   - if $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} a_n$.

6. **Limit Comparison Test.** If $a_n, b_n \geq 0$ for all n from some point on, consider the limit $c = \lim_{n\to\infty} \frac{a_n}{b_n}$.
   - if $0 < c < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together
   - if $c = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
   - if $c = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges

7. **Alternating Series Test.** If $a_n = (-1)^n b_n$ where $b_n \geq 0$, $\lim_{n\to\infty} b_n = 0$, and $b_{n+1} \leq b_n$ for all n from some point on, then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

8. **Alternating Series Estimation**. If an alternating series $\sum_{n=1} \infty (-1)^n b_n$ converges to a value S, then the error in approximating S with the N^{th} partial sum $S_N = \sum_{n=1}^{N} (-1)^n b_n$ is at most b_{n+1}:
$$|S_N - S| \leq b_{n+1}.$$

9. **Absolute Convergence.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

10. **Ratio Test.** Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

    - If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
    - If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
    - If $L = 1$, the test is inconclusive. $\sum_{n=1}^{\infty} a_n$ might converge or diverge.

11. **Root Test.** Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.

    - If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
    - If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
    - If $L = 1$, the test is inconclusive. $\sum_{n=1}^{\infty} a_n$ might converge or diverge.

12. **Telescoping Series**. Use partial fractions to rewrite the sum as a difference of simpler terms. Compute a formula for the n^{th} partial sums by determining how terms cancel, then take the limit to find the value of the series. See the example of $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ we did in class.